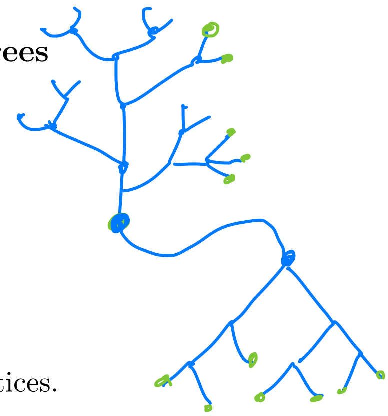
Chapter 1 - Bipartite graphs and Trees

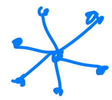


A graph is **acyclic** if it has no cycles.

A graph G is a **tree** if G is acyclic and connected.

End-vertex or **leaf** is a vertex of degree one.

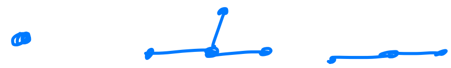
A tree is a **star** if it has exactly one vertex that are not a leaf.



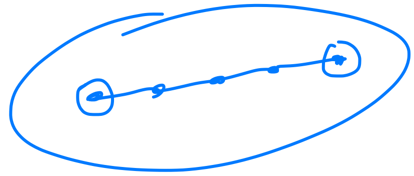
An acyclic graph is called a **forrest**.

Proposition Every tree with at least two vertices has at least two end-vertices.

1: Prove the proposition.

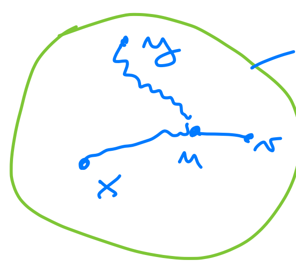


TAKE LONGEST PATH



Proposition (Tree-growing lemma) Let G be a graph and v is end-vertex. Then G is a tree if and only if $G - v$ is a tree.

2: Prove the proposition, both directions.



x-y - WALK

x-y - WALK HAS v

x ... u $\boxed{\{u v\}, v \{v u\}}$ u ... y

G TREE ⇒ G - v TREE

G - v TREE ⇒ G IS TREE

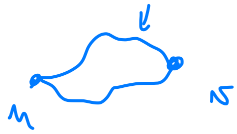
x - y WALK .. SHORTEST DOES NOT CONTAIN v

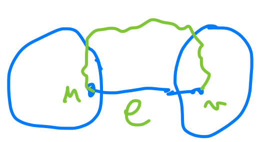
Theorem For a graph $G = (V, E)$, the following are equivalent.

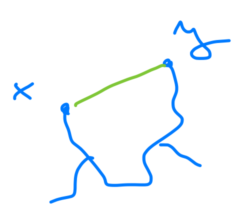
1. G is a tree.
2. (Path uniqueness) Every two vertices of G are connected by a unique path.
3. (Minimal connected) G is connected and for every edge $e \in E$, $G - e$ is disconnected.
4. (Maximal without cycle) G has no cycles and for every $x, y \in V$ such that $xy \notin E$, $G + xy$ contains a cycle.
5. (Euler's formula) G is connected and $|V| = |E| + 1$.

3: Show that 1 implies all of 2, 3, 4, 5.

CYCLE

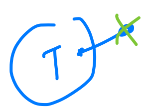






$|V| = |E| + 1$

• T_0 $1 = 0 + 1$

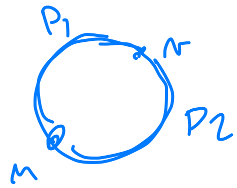


$\underbrace{|V|-1 = |E|-1+1}_{T-v}$

 by Bernard Lidický, Following Diestel, Chapter 1.5, 1.6

4: Show 2 implies 1, i.e. if every two vertices of G are connected by a unique path, then G is a tree.

CONNECTED ✓

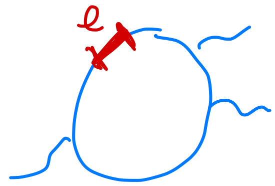


⇒ ACYCLIC

5: Show 3 implies 1, i.e. if G is connected and for every edge $e \in E$, $G - e$ is disconnected, then G is a tree.

CONNECTED ✓

⇒ G:

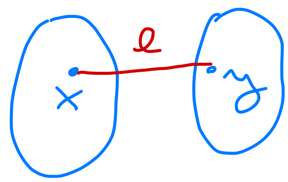


$G-e$ IS STILL CONNECTED ⇒ ↯

6: Show 4 implies 1, i.e. if G has no cycles and for every $x, y \in V$ such that $x, y \notin E$, $G + e$ contains a cycle, then G is a tree.

G ACYCLIC

G CONNECTED :



$G+e$ DOES NOT CONTAIN A CYCLE

7: Show 5 implies 1, i.e. if G is connected and $|V| = |E| + 1$, then G is a tree.
Hint: Show G has an end-vertex.

CONNECTED

$\delta(G) = 1$

$\Rightarrow \exists v : d(v) = 1$

↑

INDUCTION

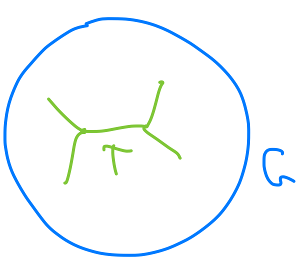
FOR CONTRADICTION $\delta(G) \geq 2$

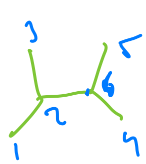
$$|E| \geq \frac{1}{2} \sum d(v) \geq \frac{1}{2} \sum_v \overset{\overset{2}{\shortparallel}}{\delta(G)} = \frac{1}{2} \sum_v 2 = \frac{1}{2} \cdot 2 \underbrace{\left(\sum_v 1\right)}_{\shortparallel \atop |V|} = |V|$$

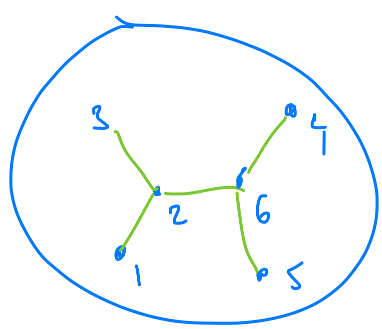
$|E| \geq |V|$ & $|V| > |E|$

Proposition 1. *If T is a tree and G is a graph with $\delta(G) \geq |T| - 1$, then $T \subseteq G$.*

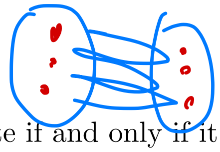
8: Prove the proposition.





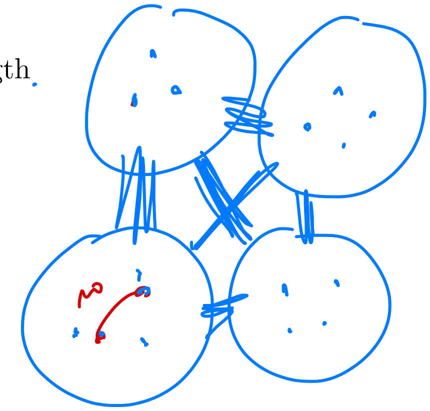


A graph $G = (V, E)$ is r-**partite** if V admits a partition into r classes such that each induces an independent set.



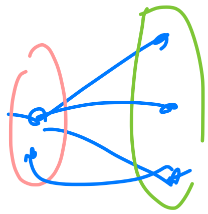
2-partite is **biartite**.

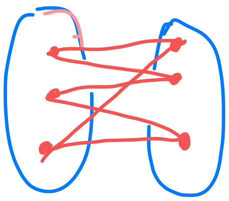
Lemma A graph is bipartite if and only if it contains no cycles of odd length.



4-PARTITE

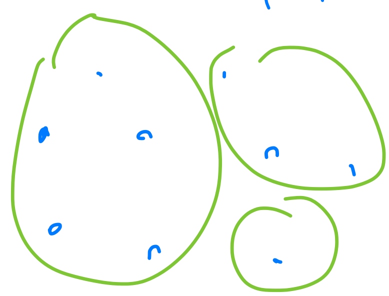
9: Prove the lemma.



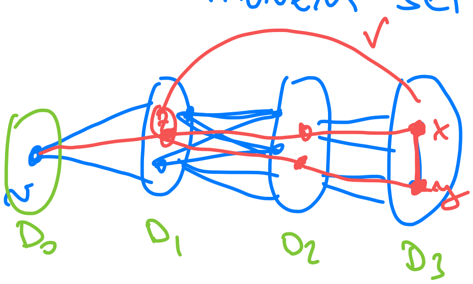


C_{2n+1} IS NOT BIPARTITE

↳ CYCLE IS EVEN



EACH CONNECTED COMPONENT SEPARATELY



$D_i :=$ VTXS IN DISTANCE i FROM v

CLAIM

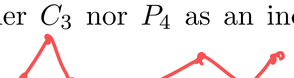
BIPARTITION: $\left(\bigcup_i D_{2i}\right) \left(\bigcup_i D_{2i+1}\right)$

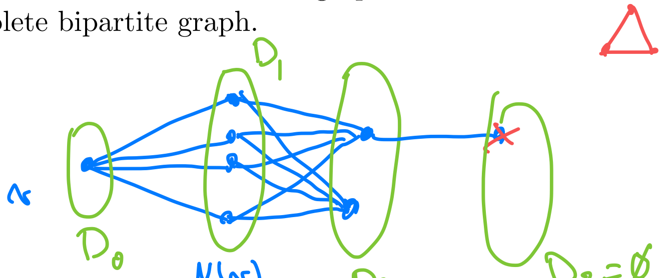
$xy \in E$

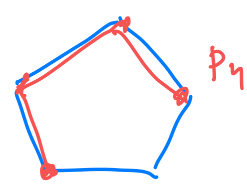
$x, y \in D_i$ $\quad$ z ON $N x \cap N y$ PATH

$z-x-y-z$ GIVES AN ODD CYCLE

10: Let G be a connected graph that has neither C_3 nor P_4 as an induced subgraph. Prove that G is a complete bipartite graph.







11: [Open problem] In a 3-regular graph, is there always a cycle whose length is a power of 2? Is it true for the Petersen's graph?

Supplemental exercises

An edge e in a connected graph G is a **bridge** if $G - e$ is not connected.

An edge e in a graph G is a **bridge** if the number of connected components in $G - e$ is more than in G.

12: An edge e of a graph G is a bridge if and only if e lies on no cycle of G.

13: List all non-isomorphic trees on 6 vertices.

A tree is G a **caterpillar** if G has at least 3 vertices and removing all leaves from G gives a path, the path is called **spine** of the caterpillar.

An acyclic graph is called a **forrest**.

14: List all non-isomorphic forests on 5 vertices.

15: Draw a star, a double star, and a caterpillar on 7 vertices. Are any of these unique?

16: Prove that every n-vertex graph with m edges has at least $m - n + 1$ cycles (different cycles, but not necessarily disjoint cycles).

17: Show that if T is a tree and $\Delta(T) = k$ then T has at least k leaves. Recall that $\Delta(T)$ means the maximum degree of T.

18: Show that sequence of natural numbers $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ is a degree sequence of some tree iff $\sum_i d_i = 2n - 2$.

19: Show that a graph G satisfying $|V| = |E| + 1$ need not be a tree.

20: [Open problem] In a 3-regular graph, is there always a cycle whose length is a power of 2? Is it true for the Petersen's graph?